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Classical Limit of an $SU(2)\times U(1)$ Gauge Field Theory for Gravity

H. Dehnen[1], F. Ghaboussi,[1] and H. Frommert[1]

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It is shown that the ground state of a Yang–Mills theory for gravity proposed by us induces in the classical limit the closed Einstein universe and its tangent space (Minkowski space), respectively.

In a previous paper we introduced a model where the linearized Einstein equations can be reached as the classical limit of a Yang–Mills-type gauge field theory of the $SU(2)\times U(1)$ spin-transformation group (Dehnen and Ghaboussi, 1985):

$$\psi' = U\psi, \qquad U = \exp(i\lambda_a(x^\alpha)\tau^a), \qquad \tau^a = \tfrac{1}{2}\sigma^0 \otimes \sigma^a \tag{1}$$

($a = 0, 1, 2, 3$). We began with a Minkowski background $\eta_{\mu\nu}$ and deduced the curve space–time metric as an effective classical field by the identification

$$g_{\mu\nu} = \eta^{ab}\omega_{\mu a}\omega_{\nu b} \tag{2}$$

with $\eta^{ab} = \text{diag}(-1, 1, 1, 1)$ as the "Minkowski"-type metric of the group space; $\omega_{\mu a}$ are the gauge potentials defined by the covariant derivative $D_\mu = \partial_\mu + ig\omega_{\mu a}\tau^a$ belonging to (1). In view of (2), the gauge coupling constant g has the dimension of a reciprocal length and the $U(1)$ potential $\omega_{\mu 0}$ determines the timelike part of the metric, whereas the $SU(2)$ potentials $\omega_{\mu i}$ ($i = 1, 2, 3$) give the spacelike part. In this sense Einstein's theory of gravity is understood as a purely classical theory resulting from a microscopic Yang–Mills structure for gravity (Dehnen and Ghaboussi, 1986).

In this connection the question arises, which metric is constructed by the *pure* gauge potentials $\overset{(0)}{\omega}_{\mu a}$ with vanishing field strength $\overset{(0)}{F}_{\mu\nu a} \equiv 0$? The inner consistency of the theory requires that the Minkowski metric $\eta_{\mu\nu}$ can

[1]Fakultät für Physik der Universität Konstanz, D-7750 Konstanz, West Germany.

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be reconstructed from (2) and it is to be expected that the Minkowski metric follows from the pure gauge potentials because $\eta_{\mu\nu}$ is also a kind of pure gauge potential with vanishing curvature tensor (Dehnen and Ghaboussi, 1987b).

In the following we show that indeed in the case of pure gauge potentials with $|\lambda_a| \ll 1$ the Minkowski background is reached. For arbitrary large gauge functions λ_a the closed Einstein universe results, with the mentioned Minkowski space-time as tangent space at every point. Thus, the Einstein universe and its tangent space (Minkowski space) correspond to the ground state of the Yang-Mills gauge field of gravity. Consequently, the deviations from the pure gauge potentials describe real gravity and induce in the classical limit deviations from the Minkowski space and the Einstein universe, respectively. It is very interesting that in this way the topology of the universe is determined uniquely by that of the 3-sphere, presumably in consequence of the compactness of the group $SU(2)$, the gauge potentials of which determine the space part of the metric.

Following this line we decompose the gauge potentials as follows:

$$\omega_{\mu a} = \overset{(0)}{\omega}_{\mu a} + A_{\mu a} \tag{3}$$

Then the metric (2) can be rewritten

$$g_{\mu\nu} = \overset{(0)}{g}_{\mu\nu} + h_{\mu\nu} \tag{4}$$

with

$$\overset{(0)}{g}_{\mu\nu} = \overset{(0)}{\omega}_{\mu a} \overset{(0)}{\omega}_{\nu}{}^{a} \tag{4a}$$

and

$$h_{\mu\nu} = \overset{(0)}{\omega}_{\mu}{}^{a} A_{\nu a} + \overset{(0)}{\omega}_{\nu}{}^{a} A_{\mu a} + A_{\mu}{}^{a} A_{\nu a} \tag{4b}$$

Here $\overset{(0)}{g}_{\mu\nu}$ represents the background (ground state) metric built up by the pure gauge potentials $\overset{(0)}{\omega}_{\mu a}$ and $h_{\mu\nu}$ describes the real gravity by the deviations $A_{\mu a}$ of $\omega_{\mu a}$ from $\overset{(0)}{\omega}_{\mu a}$.

For the determination of $\overset{(0)}{g}_{\mu\nu}$ we start from the general representation of $\overset{(0)}{\omega}_{\mu a}$. The $SU(2)$ part takes the form (Dehnen and Ghaboussi, 1987b)

$$\overset{(0)}{\omega}_{\mu i} = -\frac{1}{g}\left(f\lambda_{i|\mu} - p\varepsilon_{i}{}^{jk}\lambda_j\lambda_{k|\mu} + h\lambda_i\lambda^j\lambda_{j|\mu}\right) \tag{5}$$

$(i, j, k = 1, 2, 3)$ with

$$f = \frac{\sin\lambda}{\lambda}, \qquad p = \frac{1-\cos\lambda}{\lambda^2}, \qquad h = \frac{\lambda - \sin\lambda}{\lambda^3}, \qquad \lambda^2 = \lambda_i \lambda^i \tag{5a}$$

and the $U(1)$ pure gauge potential reads

$$\overset{(0)}{\omega}_{\mu 0} = -\frac{1}{g}\lambda_{0|\mu} \tag{6}$$

Inserting (5) and (6) into (4a), we obtain immediately

$$\overset{(0)}{g}_{\mu\nu} = -\frac{1}{g^2}\lambda_{0|\mu}\lambda_{0|\nu} + \frac{1}{g^2}[2p\lambda_{i|\mu}\lambda^i_{|\nu} + \lambda_{|\mu}\lambda_{|\nu}(1-2p)] \tag{7}$$

In order to find a simpler form of the metric, we substitute

$$\lambda_i(x^\alpha) = \lambda(x^\alpha) n_i(x^\alpha) \tag{8}$$

with

$$n_i n^i = 1 \qquad \text{and} \qquad n_{i|\mu} n^i = 0 \tag{8a}$$

The "pure" metric (7) is then

$$\overset{(0)}{g}_{\mu\nu} = -\frac{1}{g^2}\lambda_{0|\mu}\lambda_{0|\nu} + \frac{1}{g^2}\left(\lambda_{|\mu}\lambda_{|\nu} + 4\sin^2\frac{\lambda}{2}\, n_{i|\mu} n^i_{|\nu}\right) \tag{9}$$

With the new gauge functions

$$\begin{aligned} &\chi = \lambda/2, \qquad \vartheta = \arccos(\lambda_3/\lambda) \\ &\varphi = \operatorname{arctg}(\lambda_2/\lambda_1); \qquad T = \lambda_0/g \end{aligned} \tag{10}$$

the pure metric (9) is given by

$$\overset{(0)}{g}_{\mu\nu} = -T_{|\mu}T_{|\nu} + \left(\frac{2}{g}\right)^2 [\chi_{|\mu}\chi_{|\nu} + \sin^2\chi(\vartheta_{|\mu}\vartheta_{|\nu} + \sin^2\vartheta\, \varphi_{|\mu}\varphi_{|\nu})] \tag{11}$$

and the line element takes the final form

$$\overset{(0)}{ds^2} = -dT^2 + \left(\frac{2}{g}\right)^2 [d\chi^2 + \sin^2\chi(d\vartheta^2 + \sin^2\vartheta\, d\varphi^2)] \tag{12}$$

Here the well-known line element of the Einstein cosmos is reached in spherical coordinates, without any restriction or gauge fixing. According to (10), the gauge functions λ_i have the meaning of Cartesian space coordinates and λ_0 corresponds to the time coordinate. Furthermore, the value of the gauge coupling constant $g/2$ determines the reciprocal radius of the Einstein universe.

In the case of small gauge functions $|\lambda_a| \ll 1$ it immediately follows from (7) that

$$\overset{(0)}{g}_{\mu\nu} = -\frac{1}{g^2}\lambda_{0|\mu}\lambda_{0|\nu} + \frac{1}{g^2}\lambda_{i|\mu}\lambda^{i}{}_{|\nu} \tag{13}$$

This is the Minkowski metric $\eta_{\mu\nu}$, where λ_0/g and λ_i/g can be identified immediately with the cartesian time and space coordinates, respectively, so that the coupling constant g does not appear in future. Obviously, this is the local tangent space to the Einstein cosmos for $|\chi| \ll 1$ in (12). With $\lambda_0/g = x^0$ and $\lambda_i/g = x^i$ ($|x^\mu| \ll g^{-1}$) the pure gauge potentials (5) and (6) read in this case

$$\overset{(0)}{\omega}_{\mu a} = -\delta_\mu{}^a \tag{14}$$

Under the restriction to (13) and (14) one can show with the use of the Yang–Mills equations that the field $h_{\mu\nu}$ given by (4b) satisfies Einstein's linearized field equations when $|A_{\mu a}| \ll |\overset{(0)}{\omega}_{\mu a}| = 1$; furthermore, an infinitesimal gauge transformation induces an infinitesimal coordinate transformation of $h_{\mu\nu}$ (Dehnen and Ghaboussi, 1986). The linearized field equations for $A_{\mu a}$ are analyzed in detail by Dehnen and Ghaboussi (1987a). The exact equations for $A_{\mu a}$ and $h_{\mu\nu}$ describing the deviations from the Einstein universe (12) are under investigation.

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